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Transformation invariance of the longitudinal velocity correlation in grid-generated turbulence at high Reynolds numbers

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For grid Reynolds numbers from 12800 to 81000, the Frenkiel–Klebanoff–Huang data for approximately isotropic homogeneous grid-generated turbulence shows that the longitudinal correlation function is given by the simple empirical expression $f = [1+(r/2L)]^{-3}$, where r ($\gg 0.01M$) is the separation distance between two points in the fluid flow and $L = L(t)$ is the integral scale. It follows that the longitudinal velocity correlation $\langle u_1(\boldsymbol{x}+r\boldsymbol{e},t)\,u_1(\boldsymbol{x},t)\rangle = u^2 f$ with $\boldsymbol{e} = (1,0,0)$ is invariant under the separation-distance time-contraction transformations $r \rightarrow [r+(1-\lambda)\,2L], t \rightarrow \lambda^{\frac{5}{2}}t$ for all positive parameter values $\lambda \leqslant 1$. Conversely, if the longitudinal correlation function is prescribed to have the form $f = \mathscr{F}(r/L(t))$, then the indicated transformation invariance holds if and only if $\mathscr{F}(\xi) = (1+\frac{1}{2}\xi)^{-3}$. It is also shown that a Gaussian normal probability distribution at $t = 0$ and the Kármán–Howarth equation for all $t > 0$ are compatible with the transformation invariance and associated expression for f.

1. Introduction

It has recently been demonstrated that statistical-dynamical self-similarity must be featured in the free decay of grid-generated turbulence at high Reynolds numbers, with the experimentally established decay law $u^2 \propto t^{-\frac{6}{5}}$ and integral scale dependence $L \propto t^{\frac{2}{5}}$ following deductively and without any additive assumption from a Gaussian normal probability distribution over velocity fields at the initial instant $t = 0$ (Rosen 1985; Saffman 1967). From the prescribed initial expectation values

$$\langle u_i(\boldsymbol{x},0)\rangle = 0, \tag{1}$$

$$\langle u_i(\boldsymbol{x}+\boldsymbol{r},0)\,u_j(\boldsymbol{x},0)\rangle = c^2\left(\delta_{ij} - \nabla^{-2}\frac{\partial^2}{\partial r_i\,\partial r_j}\right)\delta^{(3)}(\boldsymbol{r}) \tag{2}$$

involving the single constant parameter c^2 with the physical units of $(\text{length})^5/(\text{time})^2$, it follows that $u^2 = (\text{numerical constant})\,c^{\frac{4}{5}}t^{-\frac{6}{5}}$ and $L = (\text{numerical constant})\,c^{\frac{2}{5}}t^{\frac{2}{5}}$. Empirically one has (e.g. Sreenivasan *et al.* 1980)

$$u^2 = 0.04\,U^{\frac{4}{5}}M^{\frac{6}{5}}t^{-\frac{6}{5}}, \tag{3}$$

$$L = 0.13\,U^{\frac{2}{5}}M^{\frac{3}{5}}t^{\frac{2}{5}}. \tag{4}$$

relations consistent with $c^2 = (\text{numerical constant})\,U^2M^3$.

r/M	0	0.10	0.20	0.30	0.40	0.60
$f[=R(r/U)]$	1	0.80	0.65	0.52	0.45	0.32
f[by (7)]	1	0.800	0.651	0.536	0.447	0.320
r/M	1	1.60	2	2.40	2.80	3.20
$f[=R(r/U)]$	0.19	0.09	0.06	0.04	0.03	0.02–0.03
f[by (7)]	0.180	0.090	0.061	0.043	0.032	0.024

TABLE 1. Comparison of experimental values for the longitudinal correlation function [figure 2 of Frenkiel *et al.* (1979) with the Taylor approximation $f = R(r/U)$] and values given by the empirical formula (7). Since the measurements were made at wind and water tunnels for which $L = 0.65M$, the integral scale L in (7) is eliminated in favour of the mesh length M

Approximately isotropic homogeneous grid-generated turbulence features $\langle u_i(\boldsymbol{x},t)\rangle = 0$ for all $t \geqslant 0$ and a two-point velocity correlation tensor of the form

$$\langle u_i(\boldsymbol{x}+\boldsymbol{r},t)\, u_j(\boldsymbol{x},t)\rangle = u^2\left[\left(f+\frac{1}{2}r\frac{\partial f}{\partial r}\right)\delta_{ij}-\frac{1}{2r}r_i r_j\frac{\partial f}{\partial r}\right], \tag{5}$$

where $u^2 = u^2(t) \equiv \frac{1}{3}\langle|\boldsymbol{u}(\boldsymbol{x},t)|^2\rangle$, $r \equiv |\boldsymbol{r}|$ and $f = f(r,t)$ is the longitudinal correlation function subject to the normalization condition $f(0,t) = 1$. By setting $\boldsymbol{r} = r\boldsymbol{e}$ with $\boldsymbol{e} \equiv (1,0,0)$, one obtains the longitudinal velocity correlation from (5) as

$$\langle u_1(\boldsymbol{x}+r\boldsymbol{e},t)\, u_1(\boldsymbol{x},t)\rangle = u^2 f. \tag{6}$$

As shown by the analysis presented below, the longitudinal velocity correlation (6) appears to manifest a transformation invariance symmetry in grid-generated turbulence at high Reynolds numbers. It is also shown in the following that a Gaussian normal probability distribution at $t = 0$ and the Kármán–Howarth equation for all $t > 0$ are compatible with the specific expression for f required by the transformation invariance of (6).

2. Empirical expression for the longitudinal correlation function and associated invariance of the longitudinal velocity correlation

For grid-generated turbulence at Reynolds numbers UM/ν from 12800 to 81000 and decay times such that $L = 0.65M$, the data reported by Frenkiel, Klebanoff & Huang (1979) are subsumed by the simple empirical expression

$$f = \left[1+\frac{r}{2L}\right]^{-3} \tag{7}$$

as shown by the comparison in table 1. Observe that formula (7) features the integral scale in a manner required by the general definition: $L = L(t) \equiv \int_0^\infty f(r,t)\,\mathrm{d}r$. It is also noteworthy that (7) is consistent with the asymptotic dependence for large r, $\lim_{r\to\infty} r^3 f(r,t) =$ (function of t), first predicted by Birkhoff (1954).

Consider the separation-distance time-contraction transformations

$$r \xrightarrow{\lambda} r' \equiv [r+(1-\lambda)\,2L(t)], \tag{8}$$

$$t \xrightarrow{\lambda} t' \equiv \lambda^{\frac{5}{2}} t, \tag{9}$$

in which λ is a disposable semigroup parameter in the range $0 < \lambda \leqslant 1$ and the integral scale $L = L(t)$ is given by (4). It is easy to demonstrate that the set of transformations (8), (9) for all positive $\lambda \leqslant 1$ constitutes a closed semigroup as a consequence of the relation $L(t') = \lambda L(t)$;

$$\left.\begin{array}{l} r \xrightarrow{\lambda} r' \xrightarrow{\lambda'} r'' \\ t \xrightarrow{\lambda} t' \xrightarrow{\lambda'} t'' \end{array}\right\} \Rightarrow \begin{array}{l} r \xrightarrow{\lambda\lambda'} r'', \\ t \xrightarrow{\lambda\lambda'} t''. \end{array} \tag{10}$$

Under a transformation prescribed by (8), (9), the induced mapping of the quantities (3) and (7) is given by

$$(u^2)' \equiv u^2(t') = \lambda^{-3} u^2(t), \tag{11}$$

$$f' \equiv \left[1 + \frac{r'}{2L(t')}\right]^{-3} = \lambda^3 f, \tag{12}$$

where $L(t') = \lambda L(t)$ is employed to obtain the final member of (12). Hence, the longitudinal velocity correlation (6) is invariant under (8), (9):

$$(u^2 f)' = u^2 f \tag{13}$$

or equivalently

$$\langle u_1(\boldsymbol{x} + r'\boldsymbol{e}, t')\, u_1(\boldsymbol{x}, t')\rangle = \langle u_1(\boldsymbol{x} + r\boldsymbol{e}, t)\, u_1(\boldsymbol{x}, t)\rangle. \tag{14}$$

Conversely, if the longitudinal correlation function is prescribed to have the form $f = \mathscr{F}(r/L(t))$ subject to the normalization condition $\mathscr{F}(0) = 1$, then the scaling property (12) (implied by (13) and (11)) requires

$$\mathscr{F}' \equiv \mathscr{F}\left(\frac{r'}{L(t')}\right) = \mathscr{F}\left(\frac{r + (1-\lambda)\,2L}{\lambda L}\right) = \lambda^3 \mathscr{F}\left(\frac{r}{L}\right)$$
$$\text{for all positive } \lambda \leqslant 1, \tag{15}$$

and (15) implies the empirical expression (7), i.e. $\mathscr{F}(\xi) = (1 + \tfrac{1}{2}\xi)^{-3}$. Therefore the longitudinal velocity correlation invariance property (14) is essentially equivalent to the validity of expression (7) for the longitudinal correlation function with u^2 and L given by (3) and (4).

3. Dynamical consistency

As $t \to 0$ the two-point velocity correlation tensor (5) must tend to the initial expectation value (2). Since the right-hand sides of (5) and (2) are manifestly symmetric solenoidal tensors with respect to $\boldsymbol{r}$, the $t \to 0$ correspondence obtains if and only if the trace of (5) with (7),

$$\langle \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{r}, t) \cdot \boldsymbol{u}(\boldsymbol{x}, t)\rangle = u^2\left(3f + r\frac{\partial f}{\partial r}\right) = 3u^2\left[1 + \frac{r}{2L}\right]^{-4}, \tag{16}$$

converges to the trace of (2),

$$\langle \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{r}, 0) \cdot \boldsymbol{u}(\boldsymbol{x}, 0)\rangle = 2c^2 \delta^{(3)}(\boldsymbol{r}). \tag{17}$$

It is easily verified that such a $t \to 0$ representation of the three-dimensional Dirac δ-function is indeed valid,

$$\lim_{t \to 0} \left\{3u^2\left[1 + \frac{r}{2L}\right]^{-4}\right\} = 2c^2 \delta^{(3)}(\boldsymbol{r}), \tag{18}$$

where u^2 and L are given by (3) and (4). The $t \to 0$ limit relation (18) fixes c^2 in terms of U and M as

$$c^2 = 4.4 \times 10^{-3} U^2 M^3, \tag{19}$$

and thus (3) and (4) are expressible directly in terms of c as $u^2 = 0.35 c^{\frac{4}{5}} t^{-\frac{6}{5}}$ and $L = (2.6)^{-1} c^{\frac{2}{5}} t^{\frac{2}{5}}$.

To check the dynamical consistency of (7) for $t > 0$, consider the Kármán–Howarth equation (e.g. see Batchelor 1953)

$$\frac{\partial}{\partial t}(u^2 f) - 2\nu u^2 \left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r} \frac{\partial f}{\partial r} \right) = K, \tag{20}$$

with the inertial-transfer term $K = K(r, t)$ defined by

$$K = 2r^{-2} \langle [\boldsymbol{r} \cdot \boldsymbol{u}(\boldsymbol{x}, t)] [\boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{r}, t)] \rangle. \tag{21}$$

By making the successive replacements $\boldsymbol{r} \to -\boldsymbol{r}$ and $\boldsymbol{x} \to (\boldsymbol{x} + \boldsymbol{r})$ in (21), one gets

$$K = -2r^{-2} \langle [\boldsymbol{r} \cdot \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{r}, t)] [\boldsymbol{u}(\boldsymbol{x} + \boldsymbol{r}, t) \cdot \boldsymbol{u}(\boldsymbol{x}, t)] \rangle, \tag{22}$$

and the sum of (21) and (22) produces

$$K = \langle \xi(\boldsymbol{r}, t; \boldsymbol{x}) [\boldsymbol{u}(\boldsymbol{x} + \boldsymbol{r}, t) \cdot \boldsymbol{u}(\boldsymbol{x}, t)] \rangle, \tag{23}$$

where

$$\xi(\boldsymbol{r}, t; \boldsymbol{x}) \equiv r^{-2} \boldsymbol{r} \cdot [\boldsymbol{u}(\boldsymbol{x}, t) - \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{r}, t)] = r^{-1} [u_1(\boldsymbol{x}, t) - u_1(\boldsymbol{x} + r\boldsymbol{e}, t)] \tag{24}$$

for $\boldsymbol{r} = r\boldsymbol{e}$ with $\boldsymbol{e} = (1, 0, 0)$. In view of the expectation-value equation (16) and the structure of the final member of (24), the probability average in (23) must yield a K of the form

$$K = -\alpha u L^{-1} 3u^2 \left[1 + \frac{r}{2L} \right]^{-4} \tag{25}$$

in which the dimensionless factor α may possibly depend on r and t.

The terms in f on the left-hand side of (20) are evaluated by carrying out the differentiations on (3), (4) and (7):

$$\frac{\partial}{\partial t}(u^2 f) = -\tfrac{6}{5} t^{-1} u^2 \left[1 + \frac{r}{2L} \right]^{-4}, \tag{26}$$

$$-2\nu u^2 \left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r} \frac{\partial f}{\partial r} \right) = 12 \nu u^2 L^{-1} r^{-1} \left[1 + \frac{r}{2L} \right]^{-5}. \tag{27}$$

From the ratio of the right-hand sides of (26) and (27) it follows that the viscosity-effect term (27) is small in magnitude compared with the time-derivative term (26) for

$$\frac{r}{M} \gg \frac{10 \nu t}{ML} = \left(\frac{10 U t}{L} \right) R_M^{-1}, \tag{28}$$

where $R_M \equiv UM/\nu$ is the grid Reynolds number. Since the viscosity-effect term (27) can be dropped in (20) for values of r that satisfy (28), one obtains the inertia-effect dominant equation

$$\frac{\partial}{\partial t}(u^2 f) = K. \tag{29}$$

As a consequence of (25) and (26), (29) is indeed satisfied for all r admitted by (28) and all $t > 0$ if the proportionality factor in (25) is a numerical constant, viz.

$$\alpha = \frac{2L}{5tu} = 0.26, \tag{30}$$

where (3) and (4) are employed to evaluate the middle member of (30). Hence the form for f in (7) is compatible with the Kármán–Howarth equation (20) for values of r satisfying the magnitude condition (28). It is interesting to note that α shown in (30) is expressible as $\alpha = u^{-1}(\mathrm{d}L/\mathrm{d}t)$, and therefore the average value of (24) in (23) is simply $-\alpha u L^{-1} = -L^{-1}(\mathrm{d}L/\mathrm{d}t)$.

4. Concluding summary

Changing the distance between the two spatial points in the longitudinal velocity correlation in the indicated inhomogeneous fashion that brings in the integral scale additively, the separation-distance transformation (8) compensates the time-contraction transformation (9) in the precise manner required for the invariance of the longitudinal velocity correlation expressed by (14). Concomitantly, $f = \mathscr{F}(r/L)$ takes the form (7) if u^2 and L are given by (3) and (4). This separation-distance time-contraction invariance symmetry in grid-generated turbulence at high Reynolds numbers is rooted in the inertia-effect dominant transfer of energy on the scale specified by the magnitude condition (28). A Gaussian normal probability distribution at $t = 0$ and the Kármán–Howarth equation for all $t > 0$ are compatible with the transformation invariance (14) and associated expression for f shown in (7).

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